

# Standards for Mathematical Practice

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# Standards for mathematical practice

- Make sense of problems and persevere in solving them
- Reason abstractly and quantitatively
- Construct viable arguments and critique the reasoning of others
- Model with mathematics
- Use appropriate tools strategically
- Attend to precision
- Look for and make use of structure
- Look for and express regularity in repeated reasoning

# Reason abstractly and quantitatively

Mathematically proficient students make sense of the quantities and their relationships in problem situations. Students bring two complementary abilities to bear on problems involving quantitative relationships: the ability to decontextualize—to abstract a given situation and represent it symbolically and manipulate the representing symbols as if they have a life of their own, without necessarily attending to their referents—and the ability to contextualize, to pause as needed during the manipulation process in order to probe into the referents for the symbols involved. Quantitative reasoning entails habits of creating a coherent representation of the problem at hand; considering the units involved; attending to the meaning of quantities, not just how to compute them; and knowing and flexibly using different properties of operations and objects.

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# Look for and make use of structure

Mathematically proficient students look closely to discern a pattern or structure. Young students, for example, might notice that three and seven more is the same amount as seven and three more, or they may sort a collection of shapes according to how many sides the shapes have. Later, students will see  $7 \times 8$  equals the well remembered  $7 \times 5 + 7 \times 3$ , in preparation for learning about the distributive property. In the expression  $x^2 + 9x + 14$ , older students can see the 14 as  $2 \times 7$  and the 9 as  $2 + 7$ . They recognize the significance of an existing line in a geometric figure and can use the strategy of drawing an auxiliary line for solving problems. They also can step back for an overview and shift perspective. They can see complicated things, such as some algebraic expressions, as single objects or as being composed of several objects. For example, they can see  $5 - 3(x - y)^2$  as 5 minus a positive number times a square and use that to realize that its value cannot be more than 5 for any real numbers  $x$  and  $y$ .

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# Look for and express regularity in repeated reasoning

Mathematically proficient students notice if calculations are repeated, and look both for general methods and for shortcuts. Upper elementary students might notice when dividing 25 by 11 that they are repeating the same calculations over and over again, and conclude they have a repeating decimal. By paying attention to the calculation of slope as they repeatedly check whether points are on the line through  $(1, 2)$  with slope 3, middle school students might abstract the equation

$$\frac{y - 2}{x - 1} = 3.$$

Noticing the regularity in the way terms cancel when expanding  $(x - 1)(x + 1)$ ,  $(x - 1)(x^2 + x + 1)$ , and  $(x - 1)(x^3 + x^2 + x + 1)$  might lead them to the general formula for the sum of a geometric series. As they work to solve a problem, mathematically proficient students maintain oversight of the process, while attending to the details. They continually evaluate the reasonableness of their intermediate results.



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- 4.NF.4 Build fractions from unit fractions by applying and extending previous understandings of operations on whole numbers.
- (a) Understand a fraction  $a/b$  as a multiple of  $1/b$ . For example, use a visual fraction model to represent  $5/4$  as the product  $5 \times (1/4)$ , recording the conclusion by the equation  $5/4 = 5 \times (1/4)$ .
  - (b) Understand a multiple of  $a/b$  as a multiple of  $1/b$ , and use this understanding to multiply a fraction by a whole number. For example, use a visual fraction model to express  $3 \times (2/5)$  as  $6 \times (1/5)$ , recognizing this product as  $6/5$ . (In general,  $n \times (a/b) = (n \times a)/b$ .)
  - (c) Solve word problems involving multiplication of a fraction by a whole number, e.g., by using visual fraction models and equations to represent the problem. For example, if each person at a party will eat  $3/8$  of a pound of roast beef, and there will be 5 people at the party, how many pounds of roast beef will be needed? Between what two whole numbers does your answer lie?

# Grouping the practice standards

1. Make sense of problems and persevere in solving them
6. Attend to precision

2. Reason abstractly and quantitatively
3. Construct viable arguments and critique the reasoning of others

4. Model with mathematics
5. Use appropriate tools strategically

7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Reasoning and explaining

Modeling and using tools

Seeing structure and generalizing